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Supersymmetric Quantum Mechanics for Inverse Square Potentials

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Inverse square potentials are studied in the context of supersymmetric quantum mechanics.

1. INTRODUCTION

In elementary particle physics, supersymmetry (SUSY) (Wess and Zumino, 1974) is a symmetry which relates half-integral spin particles (fermions) to particles with integral spin (bosons). Using the supersymmetric version of $SU(5)$ or $SO(10)$, it was possible to unify electromagnetic, weak, and strong interactions. There is no experimental evidence for the existence of new SUSY particles in nature. On the other hand, applications of SUSY have been found in nuclear and condensed matter physics as well. Applications of supersymmetry in nonrelativistic quantum mechanics was first discussed by Witten (1981). The aim of this paper is to analyze the inverse square potentials using supersymmetric quantum mechanics. First we discuss how to construct supersymmetric algebra in quantum mechanics. Introducing SUSY quantum mechanics, then we analyze inverse square potentials in that context. Supersymmetric quantum mechanics has a close relationship to supersymmetric field theory. Supersymmetric one-particle quantum mechanics serves as a model for the investigation of spontaneous breaking of supersymmetry, which is supposed to occur in supersymmetric field theories.

Now consider a one-particle quantum mechanical system in one dimension.

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Let the Hamiltonian H^0 be given,

$$H^0 = -\frac{1}{2}\frac{d^2}{dx^2} + V^0(x)$$

where without loss of generality $V^0(x)$ is chosen in such a way that the ground-state energy is zero. The ground state ψ_0 then satisfies $H^0\psi_0 = [-\frac{1}{2}\,d^2/dx^2 + V^0(x)]\psi_0 = 0$, whence

$$V^0(x) = \frac{1}{2}\frac{\psi_0''}{\psi_0}$$

and

$$H^0 = \frac{1}{2}\left(-\frac{d^2}{dx^2} + \frac{\psi_0''}{\psi_0}\right)$$

follow, and this suggests the introduction of the operators

$$Q^{\pm} = \frac{1}{\sqrt{2}}\left(\mp\frac{d}{dx} - \frac{\psi_0'}{\psi_0}\right)$$

One has

$$\begin{aligned}
2Q^{\pm}Q^{\mp} &= -\frac{d^2}{dx^2} \pm \left[\frac{d}{dx}, \frac{\psi_0'}{\psi_0}\right] + \left[\frac{\psi_0'}{\psi_0}\right]^2 \\
&= -\frac{d^2}{dx^2} \pm \frac{d}{dx}\frac{\psi_0'}{\psi_0} + \left[\frac{\psi_0'}{\psi_0}\right]^2 \\
&= -\frac{d^2}{dx^2} \pm \frac{\psi_0''}{\psi_0} + (1 \mp 1)\left[\frac{\psi_0'}{\psi_0}\right]^2
\end{aligned}$$

Defining

$$V^1(x) = V^0(x) - \frac{d}{dx}\frac{\psi_0'}{\psi_0}$$

$$H^1 = -\frac{1}{2}\frac{d^2}{dx^2} + V^1(x)$$

we see that

$$H^0 = Q^+Q^-, \qquad H^1 = Q^-Q^+$$

One refers to H^1 as the supersymmetric (SUSY) partner of H^0.

With a modified notation for the ground-state wave function ψ_0^0 of H^0, we define $\Phi = -\psi_0^{0\prime}/\psi_0^0$ and call the function Φ the superpotential.

Then

$$Q^{\pm} = \frac{1}{\sqrt{2}}\left[\mp \frac{d}{dx} + \Phi(x)\right]$$

$$V^0 = \frac{1}{2}(-\Phi' + \Phi^2)$$

$$V^1 = \frac{1}{2}(\Phi' + \Phi^2)$$

We can express the two Hamiltonians compactly by combining them into a matrix

$$H = \begin{pmatrix} H^1 & 0 \\ 0 & H^0 \end{pmatrix} = \frac{1}{2}p^2 + \frac{1}{2}\Phi^2 + \frac{1}{2}\sigma_z\Phi'$$

where

$$p = -i\frac{d}{dx} \qquad \text{and} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The part H^1 is known as the Fermi sector, and the part H^0 is called the Bose sector.

Now we define

$$N^- = \begin{pmatrix} 0 & Q^- \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad N^+ = \begin{pmatrix} 0 & 0 \\ Q^+ & 0 \end{pmatrix}$$

N^+, N^-, and H satisfy the supersymmetric algebra:

(a) $$\{N^+, N^-\} = N^+N^- + N^-N^+ = \begin{pmatrix} 0 & 0 \\ 0 & Q^+Q^- \end{pmatrix} + \begin{pmatrix} Q^-Q^+ & 0 \\ 0 & 0 \end{pmatrix}$$

$$\{N^+, N^-\} = \begin{pmatrix} Q^-Q^+ & 0 \\ 0 & Q^+Q^- \end{pmatrix} = H$$

(b) $$\{N^+, N^+\} = N^+N^+ + N^+N^+ = 2N^+N^+ = 0$$

$$\{N^-, N^-\} = N^-N^- + N^-N^- = 2N^-N^- = 0$$

(c) $$[H, N^+] = HN^+ - N^+H = (N^+N^- + N^-N^+)$$

$$\cdot N^+ - N^+\cdot(N^+N^- + N^-N^+)$$

$$\therefore [H, N^+] = N^-N^+N^+ - N^+N^+N^- = 0$$
$$(\because N^+N^+ = 0)$$
$$[H, N^-] = HN^- - N^-H = (N^+N^- + N^-N^+)$$
$$\cdot N^- - N^- \cdot (N^+N^- + N^-N^+)$$
$$\therefore [H, N^-] = N^+N^-N^- - N^-N^-N^+ = 0$$
$$(\because N^-N^- = 0)$$

Since the system satisfies supersymmetric algebra, we use the terminology SUSY quantum mechanics.

Integration of $\Phi = -\psi_0^{0\prime}/\psi_0^0$ gives

$$\psi_0^0 = N \exp\left\{-\int dx\, \Phi\right\}$$

where N follows from the normalization of ψ_0^0.

Let us discuss a few more properties.

In the case of potentials which are not harmonic, the commutator of the operators $Q^{\pm}$ is a function of x:

$$[Q^-, Q^+] = H^1 - H^0 = -\frac{d}{dx}\frac{\psi_0^{0\prime}}{\psi_0^0} = \Phi'$$

Further,

$$Q^-\psi_0^0 = \frac{1}{\sqrt{2}}\left[+\frac{d}{dx} - \frac{\psi_0^{0\prime}}{\psi_0^0}\right]\psi_0^0 = 0$$
$$\therefore \quad Q^-\psi_0^0 = 0$$

Multiplying $H^0 = Q^+Q^-$ and $H^1 = Q^-Q^+$ by Q^+ and Q^-, one further obtains the relations

$$Q^+H^1 - H^0Q^+ = 0, \qquad Q^-H^0 - H^1Q^- = 0$$

and they are equivalent to

$$[H, N^+] = 0, \qquad [H, N^-] = 0$$

respectively.

Let a state ψ_n^0 be given with eigenvalue E_n^0 of H^0,

$$Q^+Q^-\psi_n^0 = E_n^0\psi_n^0$$

Multiplying by Q^-, one finds

$$Q^- Q^+(Q^- \psi_n^0) = E_n^0(Q^- \psi_n^0)$$

Thus $Q^- \psi_n^0$ is an eigenstate of H^1 with eigenvalue E_n^0, except for the ground state ψ_0^0, by $Q^- \psi_0^0 = 0$.

Let ψ_n^1 be an eigenstate of H^1 with eigenvalue E_n^1,

$$Q^- Q^+ \psi_n^1 = E_n^1 \psi_n^1$$

Multiplication by Q^+ yields $Q^+ Q^-(Q^+ \psi_n^1) = E_n^1(Q^+ \psi_n^1)$.

Hence, $Q^+ \psi_n^1$ is an eigenfunction of H^0 with eigenvalue E_n^1.

The spectra of the two Hamiltonians can be derived from each other (Fig. 1).

Let us now consider the normalization. From

$$\int \psi_n^{1*} Q^- Q^+ \psi_n^1 \, dx = \int \psi_n^{1*} E_n^1 \psi_n^1 \, dx = E_n^1 \int \psi_n^{1*} \psi_n^1 \, dx$$

$$\frac{1}{E_n^1} \int \psi_n^{1*} Q^- Q^+ \psi_n^1 \, dx = \int \psi_n^{1*} \psi_n^1 \, dx$$

If ψ_n^1 is normalized to unity, then

$$\int \psi_n^{1*} \psi_n^1 \, dx = 1, \qquad \int \frac{1}{(E_n^1)^{1/2}} \psi_n^{1*} Q^- \frac{1}{(E_n^1)^{1/2}} Q^+ \psi_n^1 \, dx = 1$$

$$\int \frac{1}{(E_n^1)^{1/2}} (Q^+ \psi_n^1)^* \frac{1}{(E_n^1)^{1/2}} Q^+ \psi_n^1 \, dx = 1$$

Therefore $(E_n^1)^{-1/2} Q^+ \psi_n^1$ is also normalized to unity, i.e.,

$$\psi_n^0 = \frac{1}{(E_n^1)^{1/2}} Q^+ \psi_n^1$$

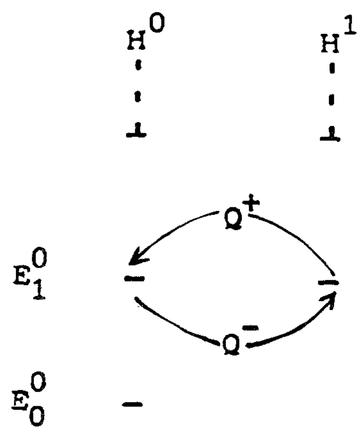

Fig. 1. Energy levels of H^0 and H^1.

An analogous relation holds for ψ_n^0:

$$\psi_n^1 = \frac{1}{(E_n^0)^{1/2}} Q^- \psi_n^0$$

2. SUSY QUANTUM MECHANICS FOR INVERSE SQUARE POTENTIALS

The radial Schrödinger equation for a two-particle system in quantum mechanics is given by

$$\frac{d^2X}{dr^2} + \left[\frac{2\mu}{\hbar^2}(E - V) - \frac{l(l+1)}{r^2}\right]X = 0$$

where μ is the reduced mass of the two-body system and $l(l+1)$ are the eigenvalues of the squared angular momentum. We have

$$\left[-\frac{1}{2}\frac{d^2}{dr^2} + \frac{\mu V}{\hbar^2} + \frac{l(l+1)}{2r^2}\right]X = \frac{\mu E}{\hbar^2} X$$

Now we consider the interactive potential as

$$V(r) = \pm\gamma/r^2, \qquad \gamma > 0$$

$$\left[-\frac{1}{2}\frac{d^2}{dr^2} + \frac{\pm\mu\gamma}{\hbar^2 r^2} + \frac{l(l+1)}{2r^2}\right]X = \frac{\mu E}{\hbar^2} X$$

$$\left[-\frac{1}{2}\frac{d^2}{dr^2} + \frac{l(l+1) \pm 2\mu\gamma/\hbar^2}{2r^2}\right]X(r) = \frac{\mu E}{\hbar^2} X(r)$$

Now take $l(l+1) \pm 2\mu\gamma/\hbar^2$ as $\nu(\nu+1)$ and $2\mu\gamma/\hbar^2$ as β. $\therefore \nu(\nu+1) = l(l+1) \pm \beta$, where $\beta = 2\mu\gamma/\hbar^2$ a positive constant. We have

$$\left[-\frac{1}{2}\frac{d^2}{dr^2} + \frac{\nu(\nu+1)}{2r^2}\right]X(r) = \frac{\mu E}{\hbar^2} X(r) \tag{1}$$

Note that $X(r) \equiv \psi(r)$.

We claim that (1) can be formulated as a problem of SUSY quantum mechanics with the operators $Q^\pm = (1/\sqrt{2})[\mp d/dr - (\nu+1)/r]$ and that the corresponding Φ is given by $\Phi = -(\nu+1)/r$, where Φ is the superpotential. This yields $\Phi^2 = (\nu+1)^2/r^2$ and $\Phi' = (\nu+1)/r^2$, and from $V^0 = \frac{1}{2}(-\Phi' + \Phi^2)$ and $V^1 = \frac{1}{2}(\Phi' + \Phi^2)$ one finds the SUSY partners

$$V^0 = \frac{1}{2}\left(\frac{(\nu + 1)^2}{r^2} - \frac{(\nu + 1)}{r^2}\right) = \frac{1}{2}\frac{\nu(\nu + 1)}{r^2}$$

$$V^1 = \frac{1}{2}\left(\frac{(\nu + 1)}{r^2} + \frac{(\nu + 1)^2}{r^2}\right) = \frac{(\nu + 1)(\nu + 2)}{2r^2}$$

We know that

$$\nu^2 + \nu - [l(l + 1) \pm \beta] = 0$$

the solutions of which are

$$\nu = -\frac{1}{2} \pm \{(l + \tfrac{1}{2})^2 \pm \beta\}^{1/2}$$

$$(\nu + 1) = \tfrac{1}{2} \pm \{(l + \tfrac{1}{2})^2 \pm \beta\}^{1/2}$$

and

$$(\nu + 2) = \tfrac{3}{2} \pm \{(l + \tfrac{1}{2})^2 \pm \beta\}^{1/2}$$

$$\therefore \quad \Phi = -\frac{1/2 \mp \{(l + 1/2)^2 \pm \beta\}^{1/2}}{r}$$

$$V^0 = \frac{l(l + 1) \pm \beta}{2r^2}$$

$$V^1 = \frac{[l(l + 1) + 1] \pm \{\beta + 2[(l + 1/2)^2 \pm \beta]^{1/2}\}}{2r^2}$$

and

$$Q^{\pm} = \frac{1}{\sqrt{2}}\left[\mp \frac{d}{dr} - \frac{1/2 \pm [(l + 1/2)^2 \pm \beta]^{1/2}}{r}\right]$$

Further, $H^0 = Q^+Q^-$ and $H^1 = Q^-Q^+$.

Thus we have succeeded in representing H^0 and H^1 as a supersymmetric pair, where

$$H^0 = -\frac{1}{2}\frac{d^2}{dr^2} + \frac{l(l + 1) \pm \beta}{2r^2}$$

and

$$H^1 = -\frac{1}{2}\frac{d^2}{dr^2} + \frac{[l(l + 1) + 1] \pm \{\beta + 2[(l + 1/2)^2 \pm \beta]^{1/2}\}}{2r^2}$$

We can recover H^0 and H^1 using

$$H = \begin{pmatrix} H^1 & 0 \\ 0 & H^0 \end{pmatrix} = \frac{1}{2}p^2 + \frac{1}{2}\Phi^2 + \frac{1}{2}\sigma_z\Phi'$$

also, where

$$p = -i\, d/dr, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hence by

$$\begin{pmatrix} H^1 & 0 \\ 0 & H^0 \end{pmatrix} = -\frac{1}{2}\frac{d^2}{dr^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\frac{(\nu+1)^2}{r^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\frac{(\nu+1)}{r^2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we obtain

$$H^1 = -\frac{1}{2}\frac{d^2}{dr^2} + \frac{(\nu+1)(\nu+2)}{2r^2}$$

$$H^0 = -\frac{1}{2}\frac{d^2}{dr^2} + \frac{\nu(\nu+1)}{2r^2}$$

We observe that $V^0(\nu + 1, r) = V^1(\nu, r)$ and $H^0(\nu + 1, r) = H^1(\nu, r)$. Using $Q^-\psi_0^0 = 0$ or $\psi_0^0 = N\exp\{-\int \Phi\, dr\}$ with $\Phi = -(\nu + 1)/r$, we can derive the ground-state wave function of H^0 corresponding to zero energy, i.e.,

$$\psi_0^0 = N\exp\left\{-\int -\frac{\nu+1}{r}\,dr\right\}$$

$$\psi_0^0 = N\exp\left\{(\nu+1)\int\frac{dr}{r}\right\} = N\exp\{(\nu+1)\log_e r\}$$

$$\psi_0^0 = N\exp\{\log_e r^{(\nu+1)}\} = N\cdot r^{\nu+1}$$

where $(\nu + 1) = \frac{1}{2} \pm \{(l+\frac{1}{2})^2 \pm \beta\}^{1/2} = \theta$ (say) when $(\nu + 1)$ is real. We have

$$\psi_0^0 = Nr^\theta$$

$$\int_0^\infty |\psi_0^0|^2\,dr = |N|^2\cdot\int_0^\infty r^{2\theta}\,dr = |N|^2\frac{r^{2\theta+1}\big|_0^\infty}{2\theta+1}$$

$$= \frac{|N|^2}{\alpha}\,r^\alpha\big|_0^\infty = \infty$$

where $\alpha = (2\theta + 1)$. $\therefore \psi_0^0$ is an unnormalizable wave function.

When the potential is attractive and $(l + 1/2)^2 < \beta$,

$$(\nu + 1) = \frac{1}{2} \pm i\delta$$

where $\delta = \{\beta - (l + 1/2)^2\}^{1/2}$. We have

$$\psi_0^0 = Nr^{(1/2 \pm i\delta)}$$
$$= Nr^{1/2} \cdot r^{\pm i\delta} = Nr^{1/2} \cdot e^{\pm i\delta \cdot \log_e r}$$
$$\therefore \int_0^\infty |\psi_0^0|^2 \, dr = |N|^2 \int_0^\infty r \, dr$$
$$= \frac{|N|^2}{2} r^2 \Big|_0^\infty = \infty$$

$\therefore \psi_0^0$ is unnormalizable.

That means there is no state corresponding to the zero ground-state energy. But supersymmetry is preserved only if such a state exists. Therefore in this case, the supersymmetry is spontaneously broken.

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